Chapter 3

Electrodynamics

3.1 Cosequence of Zero Resistance

As a consequence of zero resistance, electrons in a superconductor accelerates steadily in a constant electric field $\mathbf{E}$:

$$m\frac{d\mathbf{v_s}}{dt} = e\mathbf{E} \tag{3.1}$$

where $\mathbf{v}_s$ is the velocity of the superelectrons, m and $e = -1.602 \times 10^{-19}$C are mass and charge of superelectrons. The supercurrent density is then:

$$\mathbf{J}_s = n_s e \mathbf{v}_s \tag{3.2}$$

assuming a superelectrons density of n_s.

Combining Eq.(3.1) and Eq.(3.2), one can see that the supercurrent increases continuously with a rate given by:

$$\frac{d\mathbf{J_s}}{dt} = \frac{n_s e^2}{m}\mathbf{E}, \tag{3.3}$$

which is known as the *first London equation*.

As we known from basic electrodynamics, the electric field, current and magnetic field are correlated by Maxwell's equations:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \tag{3.4}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{3.5}$$

In a non-magnetic superconducting metal, the relative magnetic permeability $\mu_r \sim 1$ and $\mathbf{B} = \mu_0 \mathbf{H}$. The displacement current $\mathbf{D}$ is typically negligible in comparison with $\mathbf{J_s}$unless the magnetic fields vary rapidly in time. Therefore, we can write the Maxwell equations in Eq.(3.4) and Eq.(3.5) for a superconductor as:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \tag{3.6}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J_s} \tag{3.7}$$

Substituting Eq.(3.3) into Eq.(3.6):

$$\frac{\partial \mathbf{B}}{\partial t} = -\frac{m}{n_s e^2}\nabla \times \frac{\partial \mathbf{J_s}}{\partial t} \tag{3.8}$$

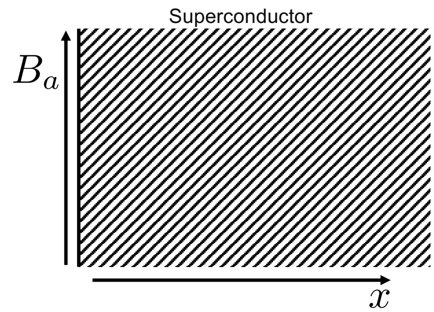

Figure 3.1: Magnetic field applied parallel to boundary of superconductor.

Replace $\mathbf{J_s}$ using Eq.(3.7) gives:

$$\frac{\partial \mathbf{B}}{\partial t} = -\frac{m}{\mu_0 n_s e^2}\nabla \times \left(\nabla \times \frac{\partial \mathbf{B}}{\partial t}\right) \tag{3.9}$$

As

$$\nabla \times \left(\nabla \times \frac{\partial \mathbf{B}}{\partial t}\right) = \nabla\left(\nabla \bullet \frac{\partial \mathbf{B}}{\partial t}\right) - \nabla^2 \frac{\partial \mathbf{B}}{\partial t} \tag{3.10}$$

and Maxwell's equations tell us

$$\nabla \bullet \mathbf{B} = 0 \tag{3.11}$$

Now Eq.(3.9) becomes

$$\nabla^2 \frac{\partial \mathbf{B}}{\partial t} = \frac{1}{\alpha}\frac{\partial \mathbf{B}}{\partial t} \tag{3.12}$$

where the constant $\alpha = \frac{m}{\mu_0 n_s e^2}$.

The magnetic flux density in a superconductor must satisfy the differential equation Eq.(3.12) as a consequence of zero resistance. Consider a infinite half superconductor with a magnetic field applied parallel to its boundary as displayed in Fig.3.1. If the applied field is uniform, Eq.(3.12) can be reduced to its one dimensional form:

$$\frac{\partial^2}{\partial x}\frac{\partial B}{\partial t} = \frac{1}{\alpha}\frac{\partial B}{\partial t} \tag{3.13}$$

with the physical solution of:

$$\frac{\partial B(x)}{\partial t} = \frac{\partial B_a}{\partial t} exp\left(\frac{-x}{\sqrt{\alpha}}\right) \tag{3.14}$$

This tells us that the changes in flux density $\frac{\partial B(x)}{\partial t}$ inside a superconductor varnish away exponentially. The magnetic flux density has a constant value in the bulk with a distance larger than $\sqrt{\alpha}$.

3.2 The London Equations

The Meissner effect says that the flux density inside a superconductor is a constant zero. Therefore not only $\frac{\partial B}{\partial t}$ but B itself must die away exponentially below the surface. To match the Meissner's experimental findings, in 1935, the London brothers F. and H. London therefore proposed that **B** also follows the differential equation Eq.(3.12) [1]:

$$\nabla^2 \mathbf{B} = \frac{1}{\alpha}\mathbf{B} \tag{3.15}$$

Similarly, the magnetic flux density dies away exponentially inside a superconductor as(Fig.3.2):

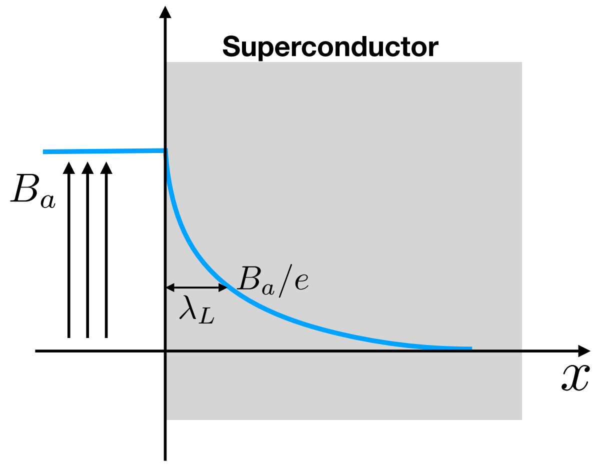

Figure 3.2: The magnetic flux density dies away exponentially inside a superconductor with a characteristic length of London penetration depth λ_L.

$$B(x) = B_a exp\left(\frac{-x}{\lambda_L}\right), \tag{3.16}$$

$$\lambda_L = \sqrt{\alpha} = \left(\frac{m}{\mu_0 n_s e^2}\right)^{1/2}. \tag{3.17}$$

The characteristic length λ_L at which the magnetic flux density drops to B_a/e is called the *London penetration depth*. The London penetration depth can be estimated ($n_s \sim 4 \times 10^{28}$ m^{-3} for typical metals) to be $\sim 10^{-8}$ m.

Equation (3.16) could be arrived naturally if one could replace $\frac{\partial B}{\partial t}$ but B starting from Eq.(3.8). Then one has the *second London equation*:

$$\mathbf{B} = -\frac{m}{n_s e^2}\nabla \times \mathbf{J_s}. \tag{3.18}$$

Together with Equation(3.3), we have derived the famous *London equations*, namely:

$$\frac{d\mathbf{J_s}}{dt} = \frac{n_s e^2}{m}\mathbf{E}, \tag{3.19}$$

$$\mathbf{B} = -\frac{m}{n_s e^2}\nabla \times \mathbf{J} = -\mu_0 \lambda^2 \nabla \times \mathbf{J}. \tag{3.20}$$

The zero resistance property of a superconductor is described by the first London equation Eq.(3.19), which essentially says that electric field is not allowed in a superconductor unless the current is changing with time. The Meissner effect is described by the second London equation, which implies that the magnetic flux density could only survive within the London penetration depth below the surface of a superconductor.

3.3 Surface Current

Consider the case of Fig.3.2, where the applied magnetic field is along the z direction. Then the Maxwell's equation Eq.(3.7) reduces to:

$$-\frac{\partial B}{\partial x} = \mu_0 j_y. \tag{3.21}$$

From Eq.(3.16), we have,

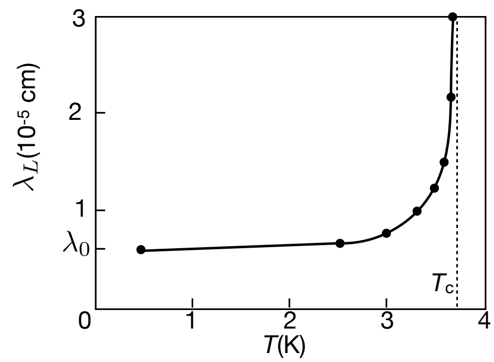

Figure 3.3: Temperature dependence of the penetration depth of Sn [2].

$$\frac{\partial B}{\partial x} = -\frac{B_a}{\lambda_L} exp(-x/\lambda_L). \tag{3.22}$$

Substitute Eq.(3.22) into Eq.(3.21):

$$j_y = \frac{B_a}{\mu_0 \lambda_L} exp(-x/\lambda_L). \tag{3.23}$$

Clearly, the current could only flow on the surface of a superconductor with the characteristic thickness of the London penetration depth.

The penetration depth is temperaure dependent and follows an emperical relation:

$$\lambda(T) = \lambda_0 \left[1 - \left(\frac{T}{T_c} \right)^4 \right]^{-1/2}, \tag{3.24}$$

where $\lambda_0$is the penetration depth at zero temperature. As shown in Fig. 3.3, the penetration depth diverges when T_c is approached from below.

Bibliography

[1] F. and H. London. The electromagnetic equations of the supraconductor. *Proceedings of the Royal Society of London. Series A - Mathematical and Physical Sciences*, 149(866):71–88, 03 1935.

[2] A. L. Schawlow and G. E. Devlin. Effect of the energy gap on the penetration depth of superconductors. *Phys. Rev.*, 113:120–126, Jan 1959.